

THE PRINCIPLE OF MATERIAL FRAME-INDIFFERENCE AND CYLINDRICAL COUETTE FLOW OF A RARIFIED GAS*

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The problem of a free molecular cylindrical Couette flow in a rotating coordinate system at different temperatures and the same angular velocities of the walls is considered. It is proved that, in an approximation which is linear in the angular velocities, the azimuthal thermal flux is proportional to the angular velocities of the walls in a fixed system of coordinates. An example of the flow of a gas which does not satisfy the principle of material frame-indifference (PMI) in the axiomatic theory of the mechanics of continuous media is thereby constructed and, consequently, the above-mentioned principle is not a universal principle of mechanics.

According to the principle of material frame-indifference, the pressure tensor P and the thermal flux vector q must be invariant under Euclidean transformations of the coordinate system such as, for example, on passing from an inertial system Σ to a Σ^* -system which rotates relative to it. In particular, P and q in a gas must not explicitly depend on its angular velocity of rotation. The PMI is a postulate concerning the existence of a higher symmetry than symmetry with respect to Galilean transformations and, hence, the substantiation of the PMI requires additional considerations and assumptions.

The results of the kinetic theory of gases which have given rise to a lengthy discussion in the literature, a critical review of which is given in /1/ (see, also /2-4/), contradict this postulate. In fact, according to the Barnett approximation P^* and q^* in the Σ^* -system not only depend on the gas-dynamic variables and their derivatives, as must be the case by virtue of the PMI, but also on the quantity $\Omega^* - W^*$ which is equal to the rotation tensor of the gas Ω in the inertial system of coordinates. (Here /1, 5/, W^* is the rotation (spin) tensor of the system Σ relative to Σ^* such that $W^*u^* = -\omega_0 \times u^*$ where u is the velocity of the gas, ω_0 is the angular velocity of the Σ^* -system relative to Σ and quantities in the Σ^* -system are indicated by using the asterisk, superscript while those in the Σ -system are without them /1/).

The property of P and q which is being considered is valid for all of the approximations of the Chapman-Enskog method with the exception of the Navier-Stokes approximation /1/. The quantity $\Omega^* - W^*$ is invariant under Euclidean transformations since it is equal to Ω and it was therefore proposed /6, 4/ that the tensor W^* should be introduced as one of the determining parameters with the aim of widening the domain of applicability of the PMI and the thermodynamics of irreversible processes in general.

However, the results which have been outlined gave rise to criticism caused by a lack of confidence in the higher approximations (starting with the Barnett approximation) of the Chapman-Enskog method for solving the Boltzmann kinetic equation (BE) for a Knudsen number $Kn \rightarrow 0$. In the opinion of several authors /7, 8/ it is necessary to obtain the solution of the boundary value problem for the BE in the Σ^* -system when $Kn \neq 0$ in order to solve this problem. It is obvious that this a difficult problem. However, a simplification of the problem exists and it follows from the fact that the effects of invariance are due to the convective terms of the BE. The collision integral in the BE is invariant to Euclidean transformations. Hence, the action of these effects must also take place under a free molecular regime, that is, in gas flows with $Kn \gg 1$ when the effect of intermolecular collisions and, consequently, the collision integral can be neglected. The results of free molecule theory have been comprehensively confirmed by calculation and experiment and are qualitatively valid over a wide range of Kn values /9, 10/.

As the simplest example, the solution of the problem of free molecule cylindrical Couette flow with different wall temperatures is presented below. This problem was chosen on account of the fact that, according to the Barnett approximation, an azimuthal thermal flux q_ϕ exists in such a flow when $Kn \ll 1$, the magnitude of which is determined by Ω . The question of the existence of such a q_ϕ was one of the fundamentals in the critical discussion mentioned above (see /1, 5/). It is proved that q_ϕ exists when $Kn \gg 1$ with the same properties under Euclidean transformations as in the Barnett approximation.

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1. Let us consider the planar motion of a gas between two infinite circular coaxial cylindrical surfaces (Fig.1) which rotate about a common axis of symmetry z with the same angular velocities. Let r_1 and r_2 be the radii of the internal and external surfaces and T_1 and T_2 be their temperatures. In the fixed Σ -system of coordinates, the walls rotate with an angular velocity ω while, in the non-inertial Σ^* -system, they rotate with an angular velocity ω^* so that

$$\omega = \omega^* + \omega_0 \quad (1.1)$$

where ω_0 is the angular velocity of rotation of the Σ^* -system relative to Σ .

The BE for a free molecule regime in a cylindrical rotating system of coordinates takes the form /1, 9, 10/

$$r \xi_r^* \frac{\partial f^*}{\partial r} + (\xi_\varphi^{*2} + 2\omega_0 r \xi_r^* + r^2 \omega_0^2) \frac{\partial f^*}{\partial \xi_r^*} - \xi_\varphi^* (\xi_r^* + 2\omega_0 r) \frac{\partial f^*}{\partial \xi_\varphi^*} = 0 \quad (1.2)$$

In (1.2), account has been taken of the fact that, in the problem under consideration, the distribution function f^* depends on the components of the velocities of the molecules ξ_r^* , ξ_φ^* , ξ_z^* and the radius $r \in [r_1, r_2]$ but is independent of z and the polar angle φ . The terms which are proportional to ω_0 are due to Coriolis effects while those in ω_0^2 involve accelerations towards the axis.

In formulating the boundary conditions we shall confine ourselves to the consideration of the commonly employed law of diffusion for the reflection of molecules from the walls when their velocities in the coordinate system associated with a given element of the surface have a Maxwell distribution. Then, the distribution functions of the reflected molecules on the internal and external surfaces are

$$f_{1,2}^v = n_{1,2} (h_{1,2}/\pi)^{3/2} \exp \{-h_{1,2} [\xi_z^{*2} + \xi_r^{*2} + (\xi_\varphi^* - \omega^* r_{1,2})^2]\} \quad (1.3)$$

$$h_{1,2} = m/(2kT_{1,2})$$

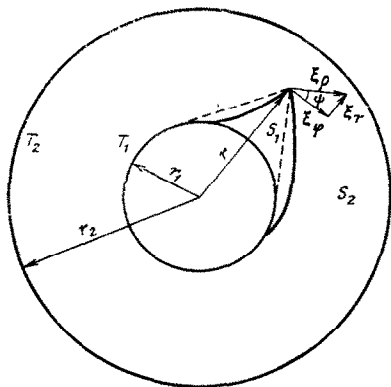


Fig.1

Here m is the mass of a molecule, k is Boltzmann's constant and the factors n_1 and n_2 are determined from the solution of the problem.

Relationship (1.2) is a linear first-order partial differential equation. The solution of the equations of its characteristics has the form

$$\xi_\varphi^* r + \omega_0 r^2 = C_1, \quad \xi_p^{*2} - \omega_0^2 r^2 = C_2, \quad (\xi_p^* = \xi_r^* + \xi_\varphi^*) \quad (1.4)$$

By virtue of (1.4) the general solution of Eq. (1.2) will be $f^* = f(\xi_z^*, C_1, C_2)$. Let us seek a particular solution in the form

$$f^* = f(\xi_z^{*2} + \xi_p^{*2} - \omega_0^2 r^2 - 2\omega_0 C_1 + C_3)$$

where C_3 is an arbitrary constant. By satisfying the boundary conditions (1.3), we finally obtain the solution of the problem in the form of a "two-sided" Maxwellian with domains of influence $S_1, S_2 = 2\pi - S_1$:

$$f^* = f_1(\xi_p^* \in S_1), \quad f^* = f_2(\xi_p^* \in S_2) \quad (1.5)$$

$$f_{1,2} = A_{1,2} \exp \{-h_{1,2} [\xi_z^{*2} + \xi_r^{*2} + (\xi_\varphi^* - \omega^* r)^2]\} \quad (1.6)$$

$$A_{1,2} = n_{1,2} (h_{1,2}/\pi)^{3/2} \exp [h_{1,2} \omega^2 (r^2 - r_{1,2}^2)]$$

The expressions for A_1 and A_2 are written taking account of (1.1). At the point r , molecules with velocities $\xi_p^* \in S_1$ are incident on the surface of the inner cylinder while those with velocities $\xi_p^* \in S_2$ are incident on the surface of the outer cylinder.

Let us stress the difference between this and the solution of the problem in a fixed system of coordinates. In the latter case the molecules move in straight lines ($\xi_\varphi^r = C_1$) with a constant energy and the regions of influence are also naturally bounded by straight lines (the broken lines in Fig.1). In the rotating system of coordinates the trajectories of the molecules are warped by the Coriolis force and the kinetic energy $E^* \sim \xi^{*2}$ is not conserved on account of the action of the centrifugal force field /11/. The boundaries of the domains of influence are found from the problem of the intersection of the curvilinear trajectories passing through the point r with the circle $r = r_1$.

The solution of (1.5), (1.6) describes, in particular, the rotation of the gas as a solid body at a constant temperature $T = T_1$. At the same time, the distribution function

$$f^* = A_1 \exp(-h_1 c^{*2}), \quad c^* = \xi^* - i_\varphi \omega^* r \quad (1.7)$$

is an exact locally Maxwellian solution of the non-linear BE: the collision integral in it is

equal to zero and f^* satisfies Eq.(1.2) /9/. The factor A_1 is determined from the normalization of the problem. For example, in a system of coordinates which rotates together with a cylindrical column of gas of finite dimensions ($\omega^* = 0$), A_1 is expressed in terms of the geometrical parameters of the column and the mean density of the gas in it /11/.

The solution of the BE in the Σ -system shows that, here, the distribution function of the molecules of the gas which is rotating as a solid body is given by the same formula (1.7) with c^* replaced by c . This is explained by the objectivity of the scalar f and the invariance of the inherent velocities of the molecules $c^* = c = \xi - u$.

The latter property follows simply from the transformations of the velocities:

$$\xi^* = \xi - i_{\varphi} \omega_0 r, \quad u^* = u - i_{\varphi} \omega_0 r$$

Hence, the locally Maxwellian solution of the BE under consideration which is written in terms of the characteristic velocities has the identical form in the Σ - and Σ^* -systems /3, 5/. At the same time A_1 is determined by the magnitude of ω and is separately independent of ω_0 or ω^* .

2. In order to obtain simple explicit expressions for the macroparameters in the Couette flow when $T_1 \neq T_2$ we shall confine ourselves to the approximation of small angular velocities by neglecting the quantities $O(\hbar_1^{-1/2} r_1 \omega)$ compared with unity and, in particular, the action of centrifugal forces. It is assumed that $\omega_0 = O(\omega)$, $r_2 - r_1 = O(r_1)$, $T_2 - T_1 = O(T_1)$. By linearizing (1.6), we can write the result in the form

$$\begin{aligned} f_{1,2} &= F_{1,2} (1 + \varepsilon_{1,2}) \\ F_{1,2} &= n_{1,2} (h_{1,2}/\pi)^{3/2} \exp(-h_{1,2} \xi_p^{*2}), \quad \varepsilon_{1,2} = 2h_{1,2} \xi_p^* \omega^* r \end{aligned} \quad (2.1)$$

In order to calculate the macroparameter at the point r it is necessary to know the boundaries of the domains of influence. Let us introduce the angle ψ^* by means of the relationship $\xi_p^* = \xi_p^* \cos \psi^*$. Since the magnitude of ξ_p^* is conserved along the trajectories, it is sufficient to establish the limits of integration with respect to $\psi^* \in [\psi_1^*, \psi_2^*]$ for $\xi_p^* \in S_1$ and $\psi^* \in [\psi_2^*, \psi_3^*]$ for $\xi_p^* \in S_2$ at the point r . Let us denote the current value of this angle (along trajectories) by ψ_p . It follows from (1.4) that

$$\cos \psi_p = (r/r_1) \cos \psi^* + (\omega_0/\xi_p^*) (r^2 - r_1^2), \quad \xi_p^* = \text{const} \quad (2.2)$$

Formula (2.2) describes the change in ψ_p under the action of a Coriolis force along a trajectory which passes through the point r and intersects the circle $r = r_1$. A solution of this intersection problem exists when $|\cos \psi_p| \leq 1$. By treating the cases of positive and negative values of $\cos \psi_p$ separately, we find

$$\begin{aligned} \psi_1^* &= \alpha + \beta, \quad \psi_2^* = \pi - \alpha + \beta, \quad \psi_3^* = 2\pi + \alpha + \beta \\ \alpha &= \arccos(r_1/r), \quad \beta = \omega_0 \sqrt{r^2 - r_1^2} / \xi_p^* \end{aligned} \quad (2.3)$$

In obtaining (2.3), linearization with respect to ω_0 of the corresponding general expressions in the values

$$\psi_i = \psi_i^* |_{\omega_0=0}, \quad i = 1, 2, 3 \quad (2.4)$$

which are valid in the case of the fixed coordinate system /10/ has been used.

Let us now calculate the azimuthal (tangential) thermal flux q_{φ} . By virtue of the objectivity of f and the invariance of c in the Σ^* -system, it is given by a formula of the same form as in the Σ -system /1, 5/

$$q_{\varphi}^* = \frac{m}{2} \int_{S_1+S_2} c_{\varphi}^* c^{*2} f^* d\xi_p^{*2} \quad (2.5)$$

By virtue of (2.5) it is first necessary to calculate the mean velocity of the gas u_{φ}^* in terms of which c_{φ}^* ($u_r^* = 0$) is determined. Using (2.1), (2.3) and (2.4), we have

$$\begin{aligned} nu_{\varphi}^* &= \int_{S_1+S_2} \xi_{\varphi}^* f^* d\xi_p^{*2} \approx \\ &= \int_{-\infty}^{\infty} \int_0^{\psi_1^*} \left(\int_{\psi_1^*}^{\psi_2^*} F_1 + \int_{\psi_2^*}^{\psi_3^*} F_2 + \int_{\psi_3^*}^{\psi_1^*} \varepsilon_1 F_1 + \int_{\psi_1^*}^{\psi_2^*} \varepsilon_2 F_2 \right) \xi_p^{*2} \cos \psi^* d\psi^* d\xi_p^* d\xi_z^* \end{aligned} \quad (2.6)$$

By carrying out some calculations we convince ourselves that the latter expression is of the order of the angular velocity ω . Hence, omitting the out of order terms, we substitute into (2.6) the formula for the number density n which describes the distribution of the latter in the problem of heat exchange between immobile walls, when

$$n = \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_{\psi_1}^{\psi_2} F_1 + \int_{\psi_2}^{\psi_1} F_2 \right) d\psi_p^* d\xi_p^* d\xi_z^* \quad (2.7)$$

It is for this reason that, here and below, the relationship $n_1/n_2 = \theta$, $\theta = (T_1/T_2)^{1/2}$, which is a corollary of the condition of no flow on the fixed internal cylinder, can be used. From (2.7), we have

$$n = n_1/(2\pi\gamma), \quad \gamma^{-1} = \pi - 2\alpha + (\pi + 2\alpha)\theta \quad (2.8)$$

Here n_1 is expressed in terms of θ , r_1/r_2 and the mean density over the region occupied by the gas /10/. After this, when account is taken of (1.1), we obtain

$$u_\varphi^* = u_\varphi - r\omega_0, \quad u_\varphi = r\omega [1 + \gamma(\theta - 1)\sin 2\alpha] \quad (2.9)$$

The gas rotates as a solid body when the temperatures of the walls are identical ($\theta = 1$).

The value of q_φ^* is calculated in the same manner as that of (2.6) but with the difference that, instead of the linearized $f_{1,2}$ in the integrands of (2.5), it is necessary to substitute the products $c_\varphi^* c^{*2} f_{1,2}$ which have been linearized in $\xi_\varphi^* \xi^{*2} F_{1,2}$. In the final analysis we have

$$q_\varphi^* = \omega r n k (T_1 T_2)^{1/2} \gamma^2 (1 - \theta) B \sin 2\alpha \quad (2.10)$$

$$B = 4\pi (1 + \theta) + (1 - \theta)(2\alpha + \sin 2\alpha), \quad \theta = (T_1/T_2)^{1/2}$$

Here, n , γ , and α are defined by formulae (2.8) and (2.3). Analogous calculations for the fixed system of coordinates yield the same result.

3. Hence, there is an azimuthal thermal flux (2.10) in cylindrical Couette flow when $T_1 \neq T_2$ which is proportional to the angular velocity ω of the walls relative to the first system of coordinates both in rotating as well as in inertial systems of coordinates.

Contrary to the PMI, in all the systems of coordinates q_φ is determined by the angular velocity ω of rotation of the walls with respect to the fixed system of coordinates and is separately independent of their angular velocity ω^* relative to the rotating system of coordinates. In other words, q_φ^* is determined by the invariant expression $\omega^* + \omega_0 = \omega$ or by the components of the tensor $\Omega^* - W^* = \Omega$ which depend in an obvious manner on ω^* , $-\omega_0$ and ω respectively.

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